

LOCAL CONSTANCY OF DIMENSIONS OF HECKE EIGENSPACES OF AUTOMORPHIC FORMS

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ABSTRACT. We use a method of Buzzard to study p -adic families of Hilbert modular forms and modular forms over imaginary quadratic fields. In the case of Hilbert modular forms, we get local constancy of dimensions of spaces of fixed slope and varying weight. For imaginary quadratic fields we obtain bounds independent of the weight on the dimensions of such spaces.

1. INTRODUCTION

In this paper we explore p -adic variations of automorphic forms. Serre [Se1] first presented the notion of a p -adic analytic family of modular eigenforms using p -adic Eisenstein series, and this provided the first application of p -adic families. Hida then showed the first example of families of cuspidal eigenforms. His results ([Hi3], [Hi4]), were limited to the case of ordinary modular forms, but proved instrumental in many number-theoretical applications.

There was a wait of about a decade before non-ordinary families were constructed. Using a rigid-analytic method of overconvergent modular forms (based on earlier work of Katz [Ka]), Coleman proved the existence of many families. He showed that almost every overconvergent eigenform of finite slope lives in a p -adic family. The slope of an eigenform is the p -adic valuation of its U_p -eigenvalue, and having finite slope is a vast generalization of being ordinary, i.e., to have a U_p -eigenvalue which is a p -adic unit. Coleman also showed that overconvergent modular forms of small slope are classical, which showed the existence of p -adic families of classical modular forms. Coleman's work was motivated by, and answered, a variety of questions and conjectures that Gouvea and Mazur [GM] had made based on ample numerical evidence. Coleman and Mazur [CM] organized Coleman's results (and more) in the form of a geometric object which was called the eigencurve. It is a rigid-analytic curve whose points correspond to normalized finite-slope p -adic overconvergent modular eigenforms of a fixed tame level N .

One of the questions which remained was how big the radius r of the disc corresponding to a family could be. In [GM], Gouvea and Mazur made some precise conjectures based on a lot of numerical

computations. The exact conjecture was disproved by Buzzard and Calegari but Wan [W] showed, using Coleman's theory of rigid-analytic methods, that an eigenform f of slope s should live in a family of eigenforms with radius p^{-t} , where $t = O(s^2)$. Using fairly elementary methods of group cohomology Buzzard [B1] found explicit bounds for the number of forms of slope α , weight k and level Np , independent of the weight k . In his unpublished paper [B2], he showed that forms have some kind of p -adic continuity and gets results similar to Wan. We use these methods in the case of Hilbert modular forms and modular forms over imaginary quadratic fields .

Let F be a totally real field, $[F : \mathbb{Q}] = d$, where d is even. Let $\mathbf{k} = (k_1, k_2, \dots, k_d)$. Then there is a notion of Hilbert cusp forms of weight \mathbf{k} and level \mathfrak{n} (an ideal of O_F). By Jacquet-Langlands we get a relation between this space and modular forms over a totally definite quaternion algebra D . Let us call that space $\mathcal{S}_{\mathbf{k}}^D(U)$, where U is a compact open subgroup of D_f^* , the adélisation of D .

Let p be a fixed rational prime inert in F and denote by T_p the Hecke operator on the space of automorphic forms $\mathcal{S}_{\mathbf{k}}^D(U)$. We give this space an integral structure for a ring R , where $O_{F,p} \subseteq R$ and call it $\mathcal{S}_{\mathbf{k}}^D(U, R)$. There is a description of this space in terms of $H^0(\Gamma^i(U), R)$, where the $\Gamma^i(U)$ are discrete, arithmetic subgroups of \mathbb{H}^d/F^* , and \mathbb{H} is the Hamiltonian algebra. We choose the U carefully so that the $\Gamma^i(U)$ are trivial. Let $\xi = p^{-\sum v_i T_p}$, where the v_i are scalars. Let $D(\mathbf{k}, \alpha)$ be the number of eigenvalues of slope α of ξ acting on $\mathcal{S}_{\mathbf{k}}^D(U)$. Then, one of our main results is:

Theorem 1.1. *Suppose U is such that each $\Gamma^i(U)$ is trivial. There exist constants β_1 and β_2 , depending on U such that if $\mathbf{k}, \mathbf{k}' > n(\alpha) := [(\beta_1\alpha - \beta_2)^d]$ and if $\mathbf{k} \equiv \mathbf{k}' \pmod{p^{n(\alpha)}}$, then $D(\mathbf{k}, \alpha) = D(\mathbf{k}', \alpha)$*

Let K be an imaginary quadratic field of class number one and let O be the ring of integers of K . Fix an odd rational prime p , which is inert in K .

We have an adelic definition of automorphic forms for modular forms over number fields and in our case for imaginary quadratic fields K . Let K_A (resp K_A^*) be the adele ring (resp the idele group) of K . We put $G_K = GL_2(K)$ and $G_A = GL_2(K_A)$. The center Z_A of G_A is isomorphic to K_A^* . For a unitary character χ of the idele class group K_A^*/K^* , let $L_2^0(G_K \backslash G_A, \chi)$ denote the space of measurable functions on G_A satisfying certain boundedness conditions.

We use the Eichler-Shimura relation between a subspace of forms of this type and group cohomology. Essentially, if Γ is a congruence subgroup of $SL_2(O)$, and $S_{g,g}(O) = S_n(O) \otimes S_g(O)$, where $S_g(O)$ is the g -th symmetric tensor power with an action of $GL_2(O)$ on both components, then a certain space of modular forms is isomorphic to $H^1(\Gamma, S_{g,g}(O))$. It is this cohomology group that is useful to us.

Let m be the minimal number of generators of Γ and let $D(g, \alpha)$ denote the number of eigenvalues of slope α and weight g for the T_p operator. Our main theorem is:

Theorem 1.2. *$D(g, \alpha)$ has an upper bound which is independent of the slope α and is always less than $[3m(\alpha + 1)^2/2]m$.*

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2. PRELIMINARIES

We refer the reader to the book [DS] for a more detailed explanation of classical modular forms, cusp forms, congruence subgroups and Hecke operators.

Let $M_k(SL_2(\mathbb{Z}))$ denote the space of modular forms of weight k for $SL_2(\mathbb{Z})$, and for Γ a congruence subgroup, let $M_k(\Gamma), S_k(\Gamma)$ denote the spaces of modular forms and cusp forms of weight k for Γ .

Definition. For congruence subgroups Γ_1, Γ_2 of $SL_2(\mathbb{Z})$, $\alpha \in GL_2^+(\mathbb{Q})$, the weight k Hecke operator $\Gamma_1 \alpha \Gamma_2$ operator takes functions $f \in M_k(\Gamma_1)$ to $M_k(\Gamma_2)$ (and similarly for cusp forms) by :

$$\Gamma_1 \alpha \Gamma_2 : f \mapsto \sum_j f|_k \beta_j, \text{ where } \beta_j \text{ are orbit representatives.}$$

Let p be a prime. When $\alpha = \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}$, we call the double coset operator the T_p operator.

Definition. If f is a non-zero modular form and an eigenform for T_n for all n , then it is called a Hecke eigenform. If p is a prime, then the p -adic valuation of the corresponding eigenvalue is called the slope of an eigenform.

We now define p -adic families of modular forms (note that they are not the same as p -adic modular forms).

Definition. Let $c \in \mathbb{Z}_p$ and for $r \geq 0$, let $B(c, r) = \{k \in \mathbb{Z}_p : |k - c| < r\}$. Let N be an integer prime to p . Then a p -adic family of modular forms of level N is a formal power series:

$$\sum_{n \geq 0} F_n q^n,$$

where each $F_n : B(c, r) \rightarrow \mathbb{C}_p$ is a p -adic analytic function, with the property that for all sufficiently large (rational) integers k , each $\sum F_n(k) q^n$ is the Fourier expansion of a modular form of weight k .

An example of a non-cuspidal family is the p -adic Eisenstein series $E_k^*(z) = E_k(z) - p^{k-1}E_k(pz)$.

The slope α subspaces have been of great interest for a while, and Gouvea and Mazur [GM] made some very precise conjectures about the dimensions of these spaces. Let $d(k, \alpha)$ be the dimension of the slope α subspace of the space of classical cuspidal eigenforms for the T_p operators. Then the exact conjecture (Buzzard and Calegari [BC] found a counterexample a few years ago) was:

- If $k_1, k_2 > 2\alpha + 2$,
- and $k_1 \equiv k_2 \pmod{p^n(p-1)}$
- Then, $d(k_1, \alpha) = d(k_2, \alpha)$ (this condition is called local constancy).

Our goal is to get a Gouvea-Mazur type of result in the case of Hilbert modular forms. In the case of modular forms over imaginary quadratic fields we get an upper bound. To get these results we need to define two more objects - Newton Polygons and Symmetric tensor powers - as we use them both in the next two sections.

Newton Polygon:

Let L be a finite free \mathbb{Z}_p -module equipped with a \mathbb{Z}_p -linear endomorphism ξ , so we can think of L as a $\mathbb{Z}_p[\xi]$ module. Let $\sum_{s=0}^t c_s X^{t-s}$ be the characteristic polynomial of ξ acting on $L \otimes \mathbb{Q}_p$. Then, if v_p denotes the usual p -adic valuation on \mathbb{Z}_p , we plot the points $(i, v_p(c_i))$ in \mathbb{R}^2 , for $0 \leq i \leq t$, ignoring the i for which $c_i = 0$. Let C denote the convex hull of these point. The Newton polygon of ξ on L is the lower faces of C , that is the union of the sides forming the lower of the two routes from $(0, 0)$ to $(t, v_p(c_t))$ on the boundary of C . This graph gives us information about the p -adic valuations of the eigenvalues of ξ . If the Newton polygon has a side of slope α and whose projection onto the x axis has length n , then there are precisely n eigenvectors of ξ with p -adic valuation equal to α . The exact statement is:

Theorem 2.1. *Let \bar{L} be a field which is complete with respect to a valuation v . Let $f(x) = \sum_{j=0}^d a_j x^j \in L[x]$ be a polynomial with $a_0, a_d \neq 0$. Let l be a line segment of the Newton polygon of f joining $(j, v(a_j))$ and $(h, v(a_h))$ with $j < h$. Then $f(x)$ has exactly $h - j$ roots γ in L such that $v(\gamma)$ is the negative of the slope of l .*

Symmetric tensor powers:

Let R be any commutative ring. For any R -algebra A and for $a, b \in \mathbb{Z}_{\geq 0}$, we let $S_{a,b}(A)$ denote the $M_2(R)$ -module $\text{Sym}^a(A^2)$ (the a^{th} symmetric power with $M_2(R)$ action). The action is given by $x\alpha = (\det \alpha)^b x S^a(\alpha)$. If A^2 has a natural basis e_1, e_2 , then $S_{a,b}(A)$ has a basis f_0, \dots, f_a where each $f_i = e_1^{\otimes i} \otimes e_2^{\otimes (a-i)}$

Another way to think of this action is to use the equivalence of $\text{Sym}^a(A^2)$ with the space of homogeneous polynomials of degree a in 2 variables. The M_2 action can be described as follows.

Let A be an R -algebra and consider the polynomial ring $A[x, y]$. Let $f(x, y) \in A[x, y]$. If $\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then $\alpha f(x, y) = f(ax + cy, bx + dy)$. We can also twist this action by the determinant, $\alpha f(x, y) = \det(\alpha)^b f(ax + cy, bx + dy)$. When considered as a module for $SL_2(\mathbb{R})$ we know $\det = 1$, so we'll call the g -th symmetric tensor power S_g .

Strategy

Let L be a finite, free R -module of rank t (L will correspond to a space of automorphic forms). We will define K to be a submodule of L such that $L/K \cong \bigoplus O/p^{a_i}O$ with the a_i decreasing and $a_i \leq n$. We consider the characteristic polynomial $p(x)$ of ξ acting on L and plot its Newton polygon.

We let L' be a space of forms corresponding to a different weight, and choose K' to be a submodule similar to K . Modulo a certain power of p we show that the spaces L/K and L'/K' are isomorphic which leads to congruences of the coefficients of the respective characteristic polynomial. This tells us that the Newton polygons of fixed slopes coincide which gives us local constancy of the slope α spaces.

In the case that the quotient spaces are not isomorphic, we get a lower bound for the Newton polygon associated to ξ which transforms into an upper bound for the slope α eigenspaces.

3. THE TOTALLY REAL CASE

3.1. Automorphic forms over Quaternion Algebras. In this section we'll define automorphic forms over quaternion algebras. Due to Jacquet Langlands, they correspond to a subspace of cuspidal Hilbert modular forms. The advantage is that one can work more easily with the definitions for quaternion algebras as things are finite. For a more detailed description see [T2] and [Hi1].

If F is a field, then a quaternion algebra D over F is a central, simple algebra of dimension 4 over F . Central means that F is the center of D and simple means that there are no two-sided ideals of D except for $\{0\}$ and D itself. For each embedding $\sigma : F \hookrightarrow \mathbb{R}$, we say that D is ramified at σ if $D_\sigma = D \otimes_{F, \sigma} \mathbb{R} \cong \mathbb{H}$, where \mathbb{H} is the Hamilton quaternion algebra. A totally definite quaternion algebra means that it ramifies at exactly all the infinite places. Let \mathbb{A} be the ring of adeles.

Let F be a totally real field, $[F : \mathbb{Q}] = d$, where d is even. Let K be a Galois extension of \mathbb{Q} , which splits D , with $F \subseteq K$. Fix an isomorphism $D \otimes_F K \cong M_2(K)$. Assume that D is a totally definite quaternion algebra over F , unramified at all finite places and fix O_D to be a maximal order of D .

Let $G = \text{Res}_{F/\mathbb{Q}} D^*$ be the algebraic group defined by restriction of scalars. Fix $\mathbf{k} = (k_\tau) \in \mathbb{Z}^I$ such that each component k_τ is ≥ 2 and all components have the same parity. Set $\mathbf{t} = (1, 1, \dots, 1) \in \mathbb{Z}^I$ and set $\mathbf{m} = \mathbf{k} - 2\mathbf{t}$. Also choose $\mathbf{v} \in \mathbb{Z}^I$ such that each $v_\tau \geq 0$, some $v_\tau = 0$ and $\mathbf{m} + 2\mathbf{v} = \mu\mathbf{t}$ for some $\mu \in \mathbb{Z}_{\geq 0}$.

For any R -algebra A and for $a, b \in \mathbb{Z}_{\geq 0}$, we let $S_{a,b}(A)$ denote the left $M_2(R)$ -module $\text{Sym}^a(A^2)$ (with the $M_2(R)$ action described in the previous section). If $\mathbf{k} \in \mathbb{Z}[I]$ and $\mathbf{m}, \mathbf{v}, \mu$ are as before we set $L_{\mathbf{k}} = \otimes_{\tau \in I} S_{m_\tau, v_\tau}(\mathbb{C})$. If R is a ring such that $O_{K,v} \subseteq R$, for some $v|p$, then, $L_{\mathbf{k}}(R) = \otimes_{\tau \in I} S_{m_\tau, v_\tau}(R)$.

Now, we'll define automorphic forms on these quaternion algebras.

First, we pick a prime rational p which is inert in K . Let M be the semigroup in $M_2(O_{F,p})$ consisting of matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ such that $c \equiv 0 \pmod{p}$ and $d \equiv 1 \pmod{p}$. Let $U \subseteq G_f$ be an open compact subgroup such that the projection to $G(F_p)$ lies inside M . If $u \in U$, let $u_p \in G(F_p)$ denote the image under the projection map.

Next, we define a weight \mathbf{k} operator.

If, $f : G(\mathbb{A}) \rightarrow L_{\mathbf{k}}(R)$ and $u = u_f \cdot u_\infty \in G(\mathbb{A})$ then,

$$(f|_{\mathbf{k}}u)(x) = u_\infty f(x \cdot u^{-1}), \text{ when } R = \mathbb{C}.$$

$$(f|_{\mathbf{k}}u)(x) = u_p f(x \cdot u^{-1}), \text{ when } R \text{ is an } O_{K,p}\text{-algebra.}$$

The space of automorphic forms for D , of level U and weight \mathbf{k} can be described as:

$$\mathcal{S}_{\mathbf{k}}^D(U) = \{f : D^* \setminus G(\mathbb{A}) \rightarrow L_{\mathbf{k}} \mid f|_{\mathbf{k}}u = f, \forall u \in U\} = \{f : G_f/U \rightarrow L_{\mathbf{k}} \mid f(\alpha \cdot x) = \alpha \cdot f(x), \forall \alpha \in D^*\}$$

$$\mathcal{S}_{\mathbf{k}}^D(U, R) = \{f : D^* \setminus G(\mathbb{A}) \rightarrow L_{\mathbf{k}}(R) \mid f|_{\mathbf{k}}u = f, \forall u \in U\}.$$

The purpose of introducing $\mathcal{S}_{\mathbf{k}}^D(U, R)$ is to give $\mathcal{S}_{\mathbf{k}}^D(U)$ an integral structure which allows us to think of $\mathcal{S}_{\mathbf{k}}^D(U, R)$ as $\oplus_{\gamma_i \in X(U)} (\gamma_i L_{\mathbf{k}}(R))^{D^* \cap \gamma_i U \gamma_i^{-1}}$. Thus, we see that $\mathcal{S}_{\mathbf{k}}^D(U, R)$ is an R -lattice in $\mathcal{S}_{\mathbf{k}}^D(U)$.

Let $X(U) = D^* \setminus G_f/U$. We know this is finite, so let $h = |X(U)|$ and let $\{\gamma_i\}_{i=1}^h$ be the coset representatives. So, $G_f = \coprod_{i=1}^h D^* \cdot \gamma_i \cdot U$.

Define $\widetilde{\Gamma^i(U)} := D^* \cap \gamma_i \cdot U \cdot G_\infty^D \cdot \gamma_i^{-1}$ and let $\Gamma^i(U) := \widetilde{\Gamma^i(U)} / \widetilde{\Gamma^i(U)} \cap F^*$.

We want to impose conditions on U such that the $\Gamma^i(U)$ are trivial. We know that $\widetilde{\Gamma^i(U)}$ are discrete arithmetic subgroups of G_∞ and that $\Gamma^i(U)$ are discrete in $G_{\infty,+}/F_\infty^*$. As D is a totally definite quaternion algebra $G_\infty \cong (\mathbb{H})^d$, where \mathbb{H} is the Hamiltonian algebra. So, $\Gamma^i(U)$ is discrete in $G_{\infty,+}/F_\infty^*$ which is compact. Thus, $\Gamma^i(U)$ is finite.

Let N be an ideal in O_F . Define:

$U_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \prod_q GL_2(O_{F,q}) \mid c_q \in NO_{F,q} \forall q \right\}$, where q runs over all the finite primes of F .

$$U_1(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in U_0(N) \mid a - 1 \in N \right\},$$

We cite a result by Hida [Hil] (Sec 7).

Lemma 3.1. (*Hida*)

Put $U(N) = \{x \in U_1(N) : x_N = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, d - 1 \in N\}$ for each ideal N of O_F . Let l be a prime ideal of O_F and let e be the ramification index of l over Q . Then, if $s > 2e/(l - 1)$, then $\Gamma^i(U(l^s))$ is torsion free for all i

According to this result U can be chosen such that $\Gamma^i(U)$ are torsion free for all i . Coupled with the statements above, this means that $\Gamma^i(U)$ are trivial, provided U is chosen carefully.

Hecke operators are defined in a similar fashion as in classical modular forms using the double coset decomposition. Let U, U' be open compact subgroups and $x \in G_f$. We define:

$$\xi = UxU' : \mathcal{S}_{\mathbf{k}}^D(U, R) \rightarrow \mathcal{S}_{\mathbf{k}}^D(U', R), \text{ where}$$

$$\xi : f \mapsto \sum f|_{\mathbf{k}} x_i, \text{ where } UxU = \coprod Ux_i.$$

In particular, we have the T_q operator. $T_q = [U\eta_q U]$, where $\eta_q = \begin{pmatrix} \pi_q & 0 \\ 0 & 1 \end{pmatrix}$ and $\pi_q \in F_f$ is 1 everywhere except at q , where it is a uniformiser.

3.2. Calculations. Let $n \leq k_i$.

We can think of $L_{\mathbf{k}}(R) = L_{k_1}(R) \otimes L_{k_2}(R) \otimes \dots \otimes L_{k_d}(R) \otimes \det()^{n_k}$, where $\det()^{n_k}$ accounts for the twist by determinants and $L_{k_i}(R)$ are simply the k_i th symmetric powers.

We define $W_{\mathbf{k}}^n(R)$ to be generated by the submodules $W_{k_1}^n(R) \otimes L_{k_2}(R) \otimes \dots \otimes L_{k_d}(R) \otimes \det()^{n_k}$, $L_{k_1}(R) \otimes W_{k_2}^n(R) \otimes \dots \otimes L_{k_d}(R) \otimes \det()^{n_k}$ and up to $L_{k_1}(R) \otimes L_{k_2}(R) \otimes \dots \otimes W_{k_d}^n(R) \otimes \det()^{n_k}$, where each $W_{k_i}^n(R)$ is generated by the $(n + 1)$ R submodules $\{p^{n-j}x^j L_{k-j}(R)\}_{j=0}^n$.

We can think of each $W_{k_i}^n(R)$ in another manner. Note that $R[x, y] = \oplus L_{k_i}(R)$. Let $J \subseteq R[x, y]$ denote the homogeneous ideal (p, x) . Then for all $n > 0$, J^n is also homogeneous, so it can be written as $J^n = \oplus W_{k_i}^n(R)$. It is not hard to check that the $W_{k_i}^n(R)$ are invariant under $\Gamma_j(U)$. The key is that if $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma^j(U)$ then $c \equiv 0 \pmod{p}$.

Case 1: $\Gamma^i(U)$ are trivial:

Thus, $\mathcal{S}_{\mathbf{k}}^D(U, R)/\mathcal{W}_{\mathbf{k}}^D(U, R) \cong \oplus_{i=1}^h L_{\vec{k}}^h(R)/\oplus_{i=1}^h W_{\mathbf{k}}^n(R)$ for all $\mathbf{k} = (k_1, k_2, \dots, k_d)$

Now, we use the fact that $L_1 \otimes L_2/C \cong L_1/W_1 \otimes L_2/W_2$, where $C = W_1 \otimes L_2 + L_1 \otimes W_2$.

Let $I_{k_i}^n(R) := L_{k_i}/W_{k_i}^n \cong \oplus_{j=1}^n O/p^j O$. This quotient depends on n .

Thus, $L_{\mathbf{k}}(R)/W_{\mathbf{k}}^n(R) \cong I_{k_1}^n \otimes \dots \otimes I_{k_d}^n = I_{\mathbf{k}}^n(R) \cong \underbrace{\oplus(O/p^n O)}_{\sigma_1 \text{ times}} \underbrace{\oplus(O/p^{n-1} O)}_{\sigma_2 \text{ times}} \dots \underbrace{\oplus(O/p^1 O)}_{\sigma_n \text{ times}}$, where

σ_i is the multiplicity of each factor.

$\cong \oplus_{i=1}^r O/p^{a_i} O$, and where $a_1 \geq a_2 \geq \dots$.

Case 2 $\Gamma^i(U)$ are not trivial:

In this case, we see $\mathcal{S}_{\mathbf{k}}^D(U, R) = \oplus_{i=1}^h (L_{\mathbf{k}}(R))^{\overline{\Gamma^i(U)}}$.

Let $\mathcal{W}_{\mathbf{k}}^D(U, R) = \{f \in \mathcal{S}_{\mathbf{k}}^D(U, R) | f(x) \in W_{\mathbf{k}}^n(R)\} \cong \oplus_{i=1}^h W_{\mathbf{k}}^n(R)^{\Gamma^i(U)}$.

Thus, $\mathcal{S}_{\mathbf{k}}^D(U, R)/\mathcal{W}_{\mathbf{k}}^D(U, R) \cong \oplus_{i=1}^r O/p^{a_i} O$ for each $\mathbf{k} = (k_1, k_2, \dots, k_d)$.

We define functions B, T based on the σ_i . These functions serve as upper and lower bounds for the Newton polygons of the Hecke operators that are central to the results.

Let $b_i = n - a_i$ and $B(j) = \sum_{i=1}^j b_i$, so we get the following formulae

$$b_i = \begin{cases} j, & \text{if } (\sum_{k=1}^j \sigma_k)h \leq i \leq (\sum_{k=1}^{j+1} \sigma_k)h, 0 \leq j \leq n \\ n, & \text{if } i \geq (\sum_{k=1}^n \sigma_k)h. \end{cases}$$

$$B(x) : \mathbb{R} \rightarrow \mathbb{R}$$

$$B(x) = \begin{cases} \sum_{k=1}^j \sigma_k(k-1) + j(x - \sum_{k=1}^j \sigma_k), & \text{if } (\sum_{k=1}^j \sigma_k)h \leq x \leq (\sum_{k=1}^{j+1} \sigma_k)h, 0 \leq j \leq n \\ \sum_{k=1}^n \sigma_k(n-1) + n(x - \sum_{k=1}^n \sigma_k), & \text{if } x \geq (\sum_{k=1}^n \sigma_k)h. \end{cases}$$

Now, let M be the smallest integer such that $2M \geq n$, let $T(x) = M + B(x-1)$. We see that T can be described as:

$$T(x+1) : \mathbb{R} \rightarrow \mathbb{R}$$

$$T(x+1) = \begin{cases} M, & \text{if } i = 0 \\ M + \sum_{k=1}^j \sigma_k(k-1) + j(x - \sum_{k=1}^j \sigma_k), & \text{if } (\sum_{k=1}^j \sigma_k)h \leq x \leq (\sum_{k=1}^{j+1} \sigma_k)h, 0 \leq j \leq n \\ M + \sum_{k=1}^n \sigma_k(n-1) + n(x - \sum_{k=1}^n \sigma_k), & \text{if } x \geq (\sum_{k=1}^n \sigma_k)h. \end{cases}$$

Let $c = \inf\{T(x)/x\}$ for $x \geq 1$.

We consider the case when the $\overline{\Gamma^i}$ are trivial, because it allows us to get a precise value for each of the σ_i in the above formula.

Since the structure of $L_{\vec{k}}(R)/W_{\vec{k}}^n(R)$ is the tensor product of d copies of $\oplus_{i=1}^n O/p^i O$, we can see that $\sigma_1 = 1.h$ as we get the n -th power only once. $\sigma_2 = (2^d - 1)h$. Carrying on we see that $\sigma_i = (i^d - (i-1)^d)h$ which gives us that $\sum_{i=1}^j \sigma_i = j^d h$. So, $B(x)$ is a piecewise linear function which has slope r for $r^d h \leq x \leq (r+1)^d h$.

Now, $\inf T(x)/x = \inf_{x \geq 0} P(x)/(x+1)$ which is at least $P(x)/2x$ for $x \geq 1$ and $P(0)/2$ for $x \leq 1$.

Let $q(x) = (x/h)^{1/d} - 1$. It follows that $q(x) < B'(x)$. If $Q(x) = \int_0^y q(y)dy$, then $Q(x) < B(x)$.

$Q(x) = (\frac{d}{d+1})(\frac{x}{h})^{(d+1)/d} - x$. Let $P(x) = M + Q(x) < T(x+1)$. So, $\frac{P(x)}{x} = \frac{M}{x} + \frac{d}{d+1}(\frac{x}{h})^{1/d} - 1$.

To find the minimum of $P(x)/x$, we use basic calculus.

$(\frac{P(x)}{x})' = 0 \Rightarrow \frac{-M}{x^2} + \frac{1}{d+1}(\frac{x}{h})^{1/d-1} = 0 \Rightarrow (\frac{x}{h})^{d+1/d} = (d+1)M$. Therefore, the minimum of $P(x)/x$ occurs at $x = h(M(d+1))^{d/(d+1)}$. Putting this value in for $P(x)/x$, we get that

$$\frac{P(x)}{x} \geq \frac{M^{1/(d+1)}}{(h(d+1))^{d/(d+1)}} + \frac{d}{(d+1)}((d+1)M)^{1/(d+1)} - 1. \text{ So, } \frac{P(x)}{x} \geq M^{1/(d+1)}(\frac{1}{d+1})^{d/(d+1)}$$

Recall that we chose M such that $2M \geq n$ which means that $\frac{P(x)}{x} > c_1 n^{1/(d+1)} - 1$, where $c_1 = (\frac{1}{d+1})^{d/(d+1)}(\frac{1}{h^{d/(d+1)}} + 1)$. Note that this is true only for $x \leq n^d(h+1)$. Since $c = \inf\{T(x)/x\}$, we see that $c = \min\{c_1 n^{1/(d+1)}, n\}$.

In the preliminaries section we outlined our strategy, where the goal was to show that the Newton polygons of certain spaces coincide. We will now define our L , K and show the exact proposition (due to Buzzard) that we use later.

Let L be a finite, free R -module of rank t (where L corresponds to $\mathcal{S}_{\vec{k}}^D(U, R)$), equipped with an R -linear endomorphism ξ . Define K to be a submodule (K is $\mathcal{W}_{\vec{k}}^D(U, R)$) such that $L/K \cong \oplus O/p^{a_i} O$ with the a_i decreasing and $a_i \leq n$. Let $b_i = n - a_i$. Let M be the smallest integer such that $2M \geq n$. Let $p(x) = \sum_{i=1}^t d_i x^i$ be the characteristic polynomial of ξ acting on L .

Then, we have the following result.

Proposition 3.2. (1) *If the above conditions hold, then the Newton polygon of ξ lies above the function B .*

(2) *If $\alpha < c$, and $K \subset L$ and $K' \subset L$ satisfy the above hypothesis, and if $L/K \cong L'/K'$ as $R[\xi]$ -modules, then the Newton Polygons of ξ of small slope will coincide.*

Proof. (1) Choose an R -basis (e_i) for L such that $(p^{a_i} e_i)$ is an R -basis for K . Let $(u_{i,j})$ be the matrix of ξ acting on L . As $\xi(K) \subseteq p^n(L)$ we get that p^{b_j} divides $u_{i,j}$.

By the definition of the characteristic polynomial $\det(x - u_{i,j}) = \sum d_s x^{t-s}$. Then, we can see that:

$$d_s = (-1)^s \sum_{J \subseteq \{1,2,\dots,t\} \text{ of size } s} \sum_{\sigma \in \text{Symm}(J)} \text{sgn}(\sigma) \prod_{j \in J} u_{j,\sigma(j)}.$$

Now, we know that $p^{b_{\sigma_j}}$ divides u_{j,σ_j} , and so we get that:

$$d_s = (-1)^s \sum_{J \subseteq \{1,2,\dots,t\} \text{ of size } s} p^{\sum_{j \in J} b_j} \sum_{\sigma \in \text{Symm}(J)} \text{sgn}(\sigma) \prod_{j \in J} u_{j,\sigma(j)} / p^{b_{\sigma(j)}}.$$

Since, the b_i are increasing and for all J of size s , $\sum_{j \in J} b_j \geq b_1 + b_2 + \dots + b_s$. This gives us d_s is divisible by $p^{B(s)}$ for all s . This means that $d_i \equiv 0 \pmod{p^{B(i)}}$ which gives us the first statement.

- (2) Let $\sum d_i x^i$ be the characteristic polynomial of ξ on L . Let L' be another free R module of rank t' , and K' a submodule such that $L'/K' \cong L/K$ and $\xi(K') \subseteq p^n L'$. Let $\sum d'_{t-i} x^i$ be the characteristic polynomial of ξ on L' . Set $d'_i = 0$ for $i > t$. Assume $t' < t$.

Claim: $d_i \equiv d'_i \pmod{p^{T(i)}}$.

Proof of claim:

Since $\sum d_{t-s} x^s = \det(x.I - (u_{j,k}))$, we can expand the coefficients d_i in terms of the matrix coefficients $u_{i,j}$ as follows.

$$d_s = (-1)^s \sum_{J \subseteq \{1,2,\dots,t\}} \sum_{\sigma \in \text{symm}(J)} \text{sgn}(\sigma) \prod_{j \in J} u_{j,\sigma(j)}, \text{ where } J \text{ is a set of size } s.$$

Let (e_i) be a basis of L such that $p^{a_i} e_i$ is a basis of K . Let f_i be the reduction of each e_i in $L/K = \overline{L}$. Since $\overline{L} \cong \overline{L'}$, choose f'_i in $\overline{L'}$ through this isomorphism and let e'_i be the lift to L of each of these f'_i . Then $p^{a_i} e'_i$ is a basis for K' . Let $(u_{j,k})$ be the matrix for ξ acting on L' . Since we know that $L'/K' \cong L/K$ we can infer that $u_{j,k} \equiv u'_{j,k} \pmod{p^{a_j}}$. We set $u'_{j,k} = 0$, if $\max\{j, k\} > t'$. To establish the claim we need to show that $\prod u_{j,k} \equiv \prod u'_{j,k} \pmod{p^{T(s)}}$. We'll show $\prod_k u_{j,k} \equiv \prod_k u'_{j,k} \pmod{p^{N_J}}$, where $N_J \geq T(s)$.

$$\begin{aligned} u_{j,k} &\equiv u'_{j,k} \pmod{p^{a_j}} \Rightarrow u_{j,k}/p^{b_k} \equiv u'_{j,k}/p^{b_k} \pmod{p^{c_{j,k}}}, \text{ where } c_{j,k} = \max\{a_j - b_k, 0\} \\ &\Rightarrow \prod_{j \in J} u_{j,\sigma(j)} / p^{b_{\sigma(j)}} \equiv \prod_{j \in J} u'_{j,\sigma(j)} / p^{b_{\sigma(j)}} \pmod{p^{\min_{j \in J} \{c_{j,\sigma(j)}\}}} \Rightarrow \prod_{j \in J} u_{j,\sigma(j)} \equiv \prod_{j \in J} u'_{j,\sigma(j)} \\ &\pmod{p^{N_J}}, \text{ where } N_J = \sum_{j \in J} b_j + \min_{j \in J} \{c_{j,\sigma(j)}\} \geq \sum_{j \in J} b_j + c_{j_0,\sigma(j_0)}, \text{ where } j_0 = \max_{j \in J} j \\ &\geq \sum_{j_0 \neq j \in J} b_j + b_{j_0} + c_{j_0,\sigma(j_0)} \geq \sum_{j_0 \neq j \in J} b_j + \max\{a_{j_0}, b_{j_0}\} \geq b_1 + b_2 + \dots + b_{s-1} + M = T(s). \end{aligned}$$

Now, the function $T(i)$ is convex, piecewise linear and $c < T(i)/i$. So, $\alpha < c \Rightarrow \alpha i < ci < T(i)$. This says that if the Newton Polygon has a side of slope $\alpha < c$, then it lies below the graph of T . The endpoints of this side are $(s_1, v_p(T(s_1)))$ and $(s_2, (T(s_2)))$. So, we see that $v_p(d_{s_i}) < T(s_i)$ for $i = 1, 2$. Now, $d_i \equiv d'_i \pmod{p^{T(i)}}$ we conclude that a Newton Polygon of this length and slope depends only on \overline{L} . So, we get that for $\alpha < c$ the number of eigenvalues of ξ on L with slope α depends only on the isomorphism class of L .

□

3.3. Lemmas. Fix a prime p in \mathbb{Q} , which is inert in K . Let $R = O_{K,p}$.

We know that $\mathcal{S}_{\mathbf{k}}^D(U, R) = \bigoplus_{i=1}^h (L_{\mathbf{k}}(R))^{\Gamma^i(U)}$.

Now, we have two cases.

- $\Gamma^i(U)$ are trivial.
- $\Gamma^i(U)$ are not trivial.

Lemma 3.3. *Let $\xi = U\eta_p U$, $U = U_1(Np)$, then $\xi(\mathcal{W}_{\mathbf{k}}^D(U, R)) \subseteq p^{n+\sum v_i} \mathcal{S}_{\mathbf{k}}^D(U, R)$, where $\eta_p = \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}$, and the v_i are constants chosen in the definition of $\mathcal{S}_{\mathbf{k}}^D(U, R)$*

Proof. First, note that if $\xi = U\eta_p U = \coprod U\eta_i$, then $(\eta_i)_p \equiv \begin{pmatrix} 0 & * \\ 0 & * \end{pmatrix} \pmod{p}$.

If $f \in \mathcal{W}_{\mathbf{k}}^D(U, R)$, then $\xi : f \mapsto \sum f|_{\mathbf{k}\eta_i}$. Now, $(f|_{\mathbf{k}\eta_i})(x) = (\eta_i)_p f(x.\eta_i^{-1})$, where $f(x.\eta_i^{-1}) \in \mathcal{W}_{\mathbf{k}}^n(U, R)$ i.e., $f(x.\eta_i^{-1}) = f_1(x_1, y_1) \otimes \dots \otimes f_d(x_d, y_d)$, where $f_i(x_i, y_i) \in W_{k_i}^n(R)$ for some i .

Under the action of a matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, where $a, c \equiv 0 \pmod{p}$, we see $\begin{pmatrix} a & b \\ c & d \end{pmatrix} f_i(x_i, y_i) = f_i(ax_i + cy_i, bx_i + dy_i)$ is divisible by p^n , because if $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ as above, then γx_i is divisible by p . So, $\gamma < p, x_i > \subseteq pR[x_i, y_i]$ which means that $\gamma < p, x_i >^n \subseteq p^n R[x_i, y_i]$.

We also need to account for the twists by the determinant factor. As we had written earlier, the action of $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ on $f(x, y)$ is given by $\det(\gamma)^{v_i} f(ax + cy, bx + dy)$, where the v_i were chosen

while defining $\mathcal{S}_{\mathbf{k}}^D(U, R)$. Since $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, where $a, c \equiv 0 \pmod{p}$, the determinant factor gives us an extra power of p . Now, in all the other factors we get p^{v_j} . Since it is a tensor product of all these factors we see that $\xi(f)$ is divisible by $p^{n+\sum v_i}$. Thus, $\xi(\mathcal{W}_{\mathbf{k}}^D(U, R)) \subseteq p^{n+\sum v_i} \mathcal{S}_{\mathbf{k}}^D(U, R)$. \square

Lemma 3.4. *Assume that $\Gamma^i(U)$ are trivial. If $\mathbf{k} \equiv \mathbf{k}' \pmod{p^{n-1}}$, then $\mathcal{S}_{\mathbf{k}}^D(U, R)/\mathcal{W}_{\mathbf{k}}^D(U, R) \cong \mathcal{S}_{\mathbf{k}'}^D(U, R)/\mathcal{W}_{\mathbf{k}'}^D(U, R)$*

Proof. We define \mathbf{k}' as follows.

$$\mathbf{k}' = \mathbf{k} + (0, 0, \dots, 0, p^{n-1}, 0, \dots, 0). \text{ i.e.}$$

$$k'_i = \begin{cases} k_i, & \text{if } i \neq i_0 \\ k_i + p^{n-1}, & \text{if } i = i_0. \end{cases}$$

Now, for each $k_i \equiv k'_i \pmod{p^{n-1}}$, we want to show that $I_{k_i}^n(R) \cong I_{k'_i}^n(R)$.

As abelian groups, each $I_{k_i}^n(R) \cong \bigoplus_{j=1}^n O/p^j O$, which means that $I_{\mathbf{k}}^n(R) \cong \bigoplus_{j=1}^r O/p^{a_j} O$.

We want to prove that as M modules $I_{\mathbf{k}}^n(R) \cong I_{\mathbf{k}'}^n(R)$ if $\mathbf{k} \equiv \mathbf{k}' \pmod{p^{n-1}}$. Let $\phi : L_{\mathbf{k}}(R) \rightarrow L_{\mathbf{k}'}(R)$, where ϕ is the identity on each component except i_0 .

On $L_{k_{i_0}}$, we can think of it as $\phi : f(x, y) \mapsto f(x, y)y_{i_0}^{p^{n-1}}$. (Note that $x = (x_1, x_2, \dots, x_d)$ and $y = (y_1, y_2, \dots, y_d)$). To show that ϕ induces a homomorphism we need to verify that

$$\phi(\gamma f(x, y)) = \gamma \phi(f(x, y)), \text{ where } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \Rightarrow \phi(\gamma f(x, y)) - \gamma \phi(f(x, y)) \in W_{\mathbf{k}+p^{n-1}}^n(R).$$

We'll show it for each i , i.e., $\phi(\gamma_i f(x_i, y_i)) - \gamma_i \phi(f(x_i, y_i)) \in \langle p, x_i \rangle^n$.

Recall that for each i , $\langle p, x_i \rangle^n = \bigoplus W_{k_i}^n$. Now, $\phi(\gamma_i f(x_i, y_i)) - \gamma_i \phi(f(x_i, y_i))$.

$$\begin{aligned} &= \phi(f(a_i x_i + c_i y_i, b_i x_i + d_i y_i)) - \begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix} \phi(f(x_i, y_i)) = (f(a_i x_i + c_i y_i, b_i x_i + d_i y_i)) y_i^{p^{n-1}} - \\ &\begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix} (f(x_i, y_i)) y_i^{p^{n-1}} = (f(a_i x_i + c_i y_i, b_i x_i + d_i y_i)) y_i^{p^{n-1}} - (f(a_i x_i + c_i y_i, b_i x_i + d_i y_i)) (b_i x_i + \\ &d_i y_i)^{p^{n-1}} \\ &= (f(a_i x_i + c_i y_i, b_i x_i + d_i y_i)) (y_i^{p^{n-1}} - (b_i x_i + d_i y_i)^{p^{n-1}}). \end{aligned}$$

It will be enough to show that $(y_i^{p^{n-1}} - (b_i x_i + d_i y_i)^{p^{n-1}}) \in \langle p, x_i \rangle^n$.

Claim:

If $f, f' \in L_{k_i}(R)$, such that $f - f' \in W_{k_i}^n$, then $f^{p^s} - f'^{p^s} \in W_{p^s k_i}^n$.

Proof of claim:

Let $f = f' + F$ for some polynomial $F \in \langle p, x_i \rangle^n$. Then $f^p = (f' + F)^p$ which means that $f^p = f'^p + G$, where $G \in (pF, F^p) \subseteq \langle p, x_i \rangle^n$. The rest of the claim follows by induction.

Now, we show that $(y_i^{p^{n-1}} - (b_i x_i + d_i y_i)^{p^{n-1}}) \in \langle p, x_i \rangle^n$. For $n = 1$, we have to know that $y_i - (b_i x_i + d_i y_i) = ((1 - d_i)y_i - b_i x_i)$. Now, $d_i \equiv 1 \pmod{p}$, so we're done. Then for $n > 1$, we use the claim with $n = 1$ and the claim with $s = n - 1, n = 1$. Let $f = y_i, f' = (b_i x_i + d_i y_i)$. By the previous step, we know that $f - f' \in \langle p, x_i \rangle \subseteq W_{k_i}^n$. So, by the claim above we know that $f^{p^s} - f'^{p^s} \in W_{p^s k_i}^n$ i.e. $(y_i^{p^{n-1}} - (b_i x_i + d_i y_i)^{p^{n-1}}) \in \langle p, x_i \rangle^n$.

□

Summary :

We know that the Newton Polygon is described in terms of $v_p(d_i)$. The proposition says that it lies above the $B(i)$ and that line segments of the polygon lying below $T(i)$ depend only on L/K . So, the functions B, T form the bottom and top boundaries of a region where the Newton Polygon depends only on the isomorphism class of L/K . Therefore, if the eigenvalues of ξ have small slope

then the side of the Newton polygon which corresponds to that slope lies in the region between B, T and, therefore, depends only on the isomorphism class of L/K . So, our main theorem based on the previous lemmas can be described below. Let c_1, c_2 be the constants obtained in the calculations in the previous section.

Theorem 3.5. *Let $D(\mathbf{k}, \alpha)$ be the number of eigenvalues of the $p^{-\sum v_i T_p}$ operator acting on $\mathcal{S}_{\mathbf{k}}^D(U, R)$. Let $\alpha \leq c_1 n^{1/(d+1)} + c_2 \Rightarrow n \geq \lfloor (\beta_1 \alpha - \beta_2)^{d+1} \rfloor = n(\alpha)$. Then, if $\mathbf{k}, \mathbf{k}' \geq n(\alpha)$, $\mathbf{k} \equiv \mathbf{k}' \pmod{p^{n(\alpha)}}$ and Γ^i are trivial, then $D(\mathbf{k}', \alpha) = D(\mathbf{k}, \alpha)$. (The β_i are constants which depend only on α and n .)*

Proof. We know from the previous lemma that if $\mathbf{k} \equiv \mathbf{k}' \pmod{p^{n-1}}$, then $\mathcal{S}_{\mathbf{k}}^D(\overline{U}, R)/\mathcal{W}_{\mathbf{k}}^D(U, R)_p \cong \mathcal{S}_{\mathbf{k}'}^D(\overline{U}, R)/\mathcal{W}_{\mathbf{k}'}^D(U, R)$. As R -modules we know that they are isomorphic to $\oplus_{i=1}^r O/p^{a_i} O$. In the section on calculations, we have defined B and T such that $\alpha < c$ means that the number of eigenvalues is the same (prop 3.2). Hence, we get that $D(\mathbf{k}, \alpha) = D(\mathbf{k}', \alpha)$. □

Note : If the Γ^i are not trivial, local constancy is not possible. It is possible to get a result similar to the one we get in the imaginary quadratic case i.e., an upper bound on $D(\vec{k}, \alpha)$ independent of the weight and depending only on the slope. Yamagami has suggested that a technique of Buzzard in his paper titled "Eigenvarieties" can be used to overcome the obstructions.

4. THE IMAGINARY QUADRATIC CASE

4.1. Preliminaries. Let K be an imaginary quadratic field of class number 1. Let O be its ring of integers and p be an odd rational prime in K , which is inert in K . In Miyake's paper [M], one has a description of automorphic forms for number fields K in terms of L_0^2 decomposition and automorphic representations. We use Taylor's definitions [T1]. Using the Eichler-Shimura isomorphism, one can translate this description of automorphic forms to cohomology groups.

For any pair of non-negative integers n_1, n_2 we have a free $(n_1 + 1)(n_2 + 1)$ -dimensional O module with an action of $GL_2(O)$ (or $M_2(O)$). It may be explicitly described as $S_{n_1}(O^2) \otimes S_{n_2}(O^2)$, where S_n denotes the n -th symmetric power, and where $\gamma \in GL_2(O)$ acts on the first O^2 in the natural fashion and on the second via $\overline{\gamma}$ (complex conjugation). We will denote this module S_{n_1, n_2} . If A is an O module $S_{n_1, n_2}(A)$ will denote $S_{n_1, n_2}(O) \otimes (A)$. In particular, $S_{n_1, n_2}(\mathbb{C})$ is an irreducible finite dimensional representation of the Lie group $SL_2(\mathbb{C})$.

We are interested in the cohomology of the congruence subgroups $\Gamma < SL_2(O)$ with coefficients in $S_{n_1, n_2}(A)$. We let Γ be a congruence subgroup of $SL_2(O)$ which is finitely generated. We refer the reader to [So] for the description.

We will consider $H^1(\Gamma, S_{n,n}(O))$ as our space of modular forms over K . There is a notion of Hecke operators associated to this space. Using Taylor's description they can be summarised as,

- $[\Gamma_2 g \Gamma_1] : M^{\Gamma_1} \rightarrow M^{\Gamma_2}$ by $m \mapsto \sum (\gamma_i g) m$
- $[\Gamma_2 g \Gamma_1] : H^1(\Gamma_1, M) \rightarrow H^1(\Gamma_2, M)$ is induced by sending a Γ_1 -cocycle ϕ to the Γ_2 -cocycle $\delta \mapsto \sum (\gamma_i g) \phi((\gamma_i g)^{-1} \delta (\gamma_{j_i} g))$, where j_i is the unique index such that $\gamma_i^{-1} \delta \gamma_{j_i} \in g \Gamma_i g^{-1}$

4.2. Calculations. Fix n and let $g \geq n$.

Let $\Gamma = \Gamma_1(Np)$, and let m be the minimal number of generators of Γ . Let $S_{g,g} = S_g(O_p^2) \otimes S_g(O_p^2)$, which was described above in the previous section. We can think of S_g in terms of homogeneous polynomials of degree g . Let $O_p[x, y] = \oplus S_g(O_p^2)$. Let $J \subseteq O_p[x, y]$ denote the homogeneous ideal (p, x) . Then for all $n > 0$, J^n is also homogeneous, so it can be written as $J^n = \oplus M_g^n(O_p)$. The $M_g^n(O_p)$ are preserved under Γ . Since if $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$ then $c \equiv 0 \pmod{p}$.

We define $W_{g,g}^n = M_g^n \otimes S_g(O_p^2) + S_g(O_p^2) \otimes M_g^n$

Let $I_{g,g}^n := S_g/M_g^n \otimes S_g/M_g^n \cong \oplus_{i=1}^{n^2} O/p^{a_i} O$, where $1 \leq a_i \leq n$ and are arranged in decreasing order. Then, we have a short exact sequence,

$$0 \rightarrow W_{g,g}^n \rightarrow S_{g,g} \rightarrow I_{g,g}^n \rightarrow 0.$$

This leads to a long exact sequence,

$$H^0(\Gamma, I_{g,g}^n) \rightarrow H^1(\Gamma, W_{g,g}^n) \rightarrow H^1(\Gamma, S_{g,g}) \rightarrow H^1(\Gamma, I_{g,g}^n) \rightarrow H^2(\Gamma, W_{g,g}^n)$$

Now, we take the maximal torsion-free quotient TF and see that,

$$0 \rightarrow H^1(\Gamma, W_{g,g}^n)^{TF} \rightarrow H^1(\Gamma, S_{g,g})^{TF} \rightarrow H^1(\Gamma, I_{g,g}^n)^* \rightarrow 0, \text{ where } H^1(\Gamma, I_{g,g}^n)^* \text{ is a subquotient of } H^1(\Gamma, I_{g,g}^n)$$

Consider, $L_g = H^1(\Gamma, S_{g,g})^{TF}$ and $K_g^n = H^1(\Gamma, W_{g,g}^n)^{TF}$.

Now, L_g/K_g^n is a subquotient of $H^1(\Gamma, I_{g,g}^n)$, which is itself a quotient of the group of 1-cocycles from Γ to $I_{g,g}^n$. This group of cocycles is isomorphic to a subgroup of $(I_{g,g}^n)^m$, where m is the minimal number of generators of Γ and $I_{g,g}^n \cong \oplus_{i=1}^{n^2} O/p^{a_i} O$. Thus, $L_g/K_g^n \simeq \oplus_{i=1}^s (O/p^{a_i} O)$, where $1 \leq s \leq mn^2$

As in the case of Hilbert modular forms, we define $b_i^g = n - a_i^g$ and $B^g(j) = \sum_{i=1}^j b_i^g$.

4.3. Lemmas. Let $\xi = T_p = \Gamma \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} \Gamma$. From [T1] we know $\Gamma \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} \Gamma = \coprod \Gamma \alpha_u$, where $\alpha_u = \begin{pmatrix} p & u \\ 0 & 1 \end{pmatrix}$ and u runs over any set of representatives for congruent class of $O \bmod p$. If ϕ is a Γ -cocycle, then it gets mapped to the cocycle $\delta \mapsto \sum (\alpha_u) \phi((\alpha_u)^{-1} \delta \alpha_{v_u})$, where v_u is the unique v such that $\alpha_u^{-1} \delta \alpha_{v_u} \in \Gamma$.

Lemma 4.1. $\xi(K_g^n) \subseteq p^n(L_g)$.

Proof. Now, if $\kappa : \Gamma \rightarrow W_{g,g}^n$ is a 1-cocycle, it will be enough to show that $\xi(\kappa)$ is divisible by p^g . By the definition of ξ , we need to show that $\begin{pmatrix} p & u \\ 0 & 1 \end{pmatrix} M_g^n \subset p^n(S_g)$. Let $f(x, y) \in M_g^n$. Then under the action of $\alpha = \begin{pmatrix} p & u \\ 0 & 1 \end{pmatrix}$, $f(x, y) \mapsto f(ax + cy, bx + dy)$. Now, $f(x, y)$ is a homogeneous polynomial of degree g , and $a, c \equiv 0 \bmod p$ so $f|_\alpha = p^g f_1(x, y)$, where $f_1(x, y) \in S_g$. Thus, we get that $\xi(K_g^n) \subseteq p^n(L_g)$. \square

Let $\xi : L \rightarrow L$ be an O_p linear endomorphism such that $\xi(K) \subseteq p^n(L)$. Let \overline{L} denote L/K with its induced action of ξ , where $L/K \cong \oplus_{i=1}^r O/p^{a_i} O$ where the $a_i \leq n$ and are decreasing and define $b_i = n - a_i$ and $B(j) = \sum_{i=1}^j b(i)$. Let $\sum_{i=1}^t d_i X^i$ be the characteristic polynomial of ξ acting on L , and write $d_s = 0$ for $s > t$. From now on, we shall think of L as an $O_p[\xi]$ module, where $O_p[\xi]$ is thought of as an indeterminate over O_p . So, K is also a $O_p[\xi]$ submodule of L .

Now, we state our key lemma.

Lemma 4.2. $d_i \equiv 0 \bmod p^{B(i)}$.

Proof. Choose an O_p basis (e_i) of L such that $(p^{a_i} e_i)$ is an O_p basis for M . Let $(u_{i,j})$ be the matrix for the action of ξ on L with respect to the basis we've chosen. The assumption that $\xi(K) \subseteq p^n(L)$ means that $(u_{i,j}) p^{a_j} e_j = p^n w$, where $w \in L$ implies that p^{b_j} divides $u_{i,j}$, since $b_j = n - a_j$.

By the definition of the characteristic polynomial $\det(x - u_{i,j}) = \sum d_s x^{t-s}$. Then, we can see that:

$$d_s = (-1)^s \sum_{J \subseteq \{1, 2, \dots, t\} \text{ of size } s} \sum_{\sigma \in \text{Sym}(J)} \text{sgn}(\sigma) \prod_{j \in J} u_{j, \sigma(j)}.$$

Since we know that $p^{b_{\sigma(j)}}$ divides $u_{j, \sigma(j)}$, we can rewrite d_s as:

$$d_s = (-1)^s \sum_{J \subseteq \{1, 2, \dots, t\} \text{ of size } s} p^{\sum_{j \in J} b_j} \sum_{\sigma \in \text{Sym}(J)} \text{sgn}(\sigma) \prod_{j \in J} u_{j, \sigma(j)} / p^{b_{\sigma(j)}}.$$

Since, the b_i are increasing and for all J of size s , $\sum_{j \in J} b_j \geq b_1 + b_2 + \dots + b_s$. Hence, d_s is divisible by $p^{B(s)}$ for all s and the lemma is proved. \square

Lemma 4.3. *Let P be a finite p group and let $p^n P = 0$. If Q is a subquotient of P then, for $0 \leq \mu \leq n - 1$,*

$$|p^\mu Q / p^{\mu+1} Q| \leq |p^\mu P / p^{\mu+1} P|$$

Let $P = Z^1(\Gamma, I_g^n) = (I_g^n)^m$. The above lemma says that if L_g/K_g^n is a subquotient of P (a finite p group), then $B^g > B$, where B is the function defined for P , and B^g is defined for L_g/K_g^n . (We defined B in the previous section). This gives us the following result:

Lemma 4.4. *The Newton polygon of ξ on L_g has a uniform lower bound. This lower bound is a piecewise linear function, which has slope 0 for $0 \leq x \leq m$, slope 1 for $m \leq x \leq 4m$, and in general slope r for $r^2 m \leq x \leq (r+1)^2 m$, $r < n$, and slope n for $x \geq n^2 m$*

Let $D(g, \alpha)$ be the number of eigenvalues of T_p acting on the space of automorphic forms of weight g , slope α . We have the following result:

Theorem 4.5. *$D(g, \alpha)$ has a uniform upper bound and is always less than $[3m(\alpha + 1)^2/2]m$.*

Proof. To get an upper bound on $D(g, \alpha)$, we consider the Newton polygon of the T_p operator on the space of modular forms of weight g . Since we know that the eigenvalues of slope α is given by the length of the projection on the x axis of that side, we find an upper bound for that projection.

By the previous lemma, we know that the Newton polygon of this operator is bounded below by the function B which has slope r for $r^2 m \leq x \leq (r+1)^2 m$. We can bound B from below by the function $(2/3)(x/m)^{3/2} - x$ (see the previous section for an explanation of how we get this lower bound). This lower bound for the Newton polygon transforms into an upper bound for the slope α subspaces, which proves the theorem. \square

5. CONCLUDING REMARKS

We remark that using Jacquet-Langlands (see [Hi2] for a detailed exposition) we can translate the above theorems to results on cuspidal Hilbert modular forms. Our T_p operator has an extra power of p due to the determinant. In the imaginary quadratic case, the torsion in the cohomology groups is an obstruction towards getting local constancy of the slope α subspaces. The author is in

the process of making some computations in the classical case to see the amount of torsion involved in the cohomology groups, which will hopefully lead to some insight in the imaginary quadratic case. For some examples of Hilbert Modular forms over real quadratic fields see Dembele's papers as well as an ongoing project to develop a database of such examples. Finally, similar results on local constancy for Hilbert modular forms have been proved by Yamagami [Y], who kindly read through a draft and offered many useful suggestions.

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